Matrices as Babushka dolls: from the Conley index to symbolic dynamics

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(Cornell REU Summer 2006, and beyond)

Overview

Given the Conley index, we want symbolic dynamics

Compute a symbolic dynamical system which is semi-conjugate

Take index map and perform various operations yielding progressively smaller matricies

This procession of “nested” matricies is a lot like...

Babushka dolls: a (perhaps) useful analogy

The dolls:

1. index map
2. index map with transient generators removed
3. induced symbol system on regions
4. verified symbol system
5. amalgamated symbol system

$1 \to 2$: Removing transient generators

- Purpose: simplify the index map representative

- Intuition: taking the ‘least common denominator’ of the shift equivalence class

- Algorithm: view the index map as a graph and remove nodes not in or between strongly-connected components

$2 \to 3$: Contract regions

Given regions $R_1, \ldots, R_n$, create a graph T:

$$V(T) = [n]$$

$$E(T) = \{(i,j) | R_i \text{ maps to } R_j\},$$

where "maps" means there are generators that map that way

Natural first guess at a symbol map σ_T

$3 \to 4$: Compute semi-conjugacy using the Conley index

A symbolic dynamical system σ_T is semi-conjugate to f if

$$\mathrm{Con}(f|_{R_{s_k}} \circ \cdots \circ f|_{R_{s_1}}) \text{ is not nilpotent}$$

for all cycles $s_1, \ldots, s_k$ in T

Problem: infinite number of computations

Solution: look for patterns

We say a path $p = (p_1, \ldots, p_k)$ reduces to a shorter path $q = (q_1, \ldots, q_{k'})$ if

$$\mathrm{Con}(f|_{R_{p_1}} \circ \cdots \circ f|_{R_{p_k}}) = \mathrm{Con}(f|_{R_{q_1}} \circ \cdots \circ f|_{R_{q_{k'}}}) \neq 0,$$

If for some k, all paths of length k reduce to shorter paths then σ_T is semi-conjugate to f

If we encounter paths yielding a trivial Conley index, we have to cut an edge of T from this path

Algorithm to compute semi-conjugacy

1. Fix $k \approx 30$

2. Collect ‘bad edge sets’: edges of paths that

   (a) have length at most k and have trivial Conley index

   (b) are of length k and do not reduce (assume the worst)

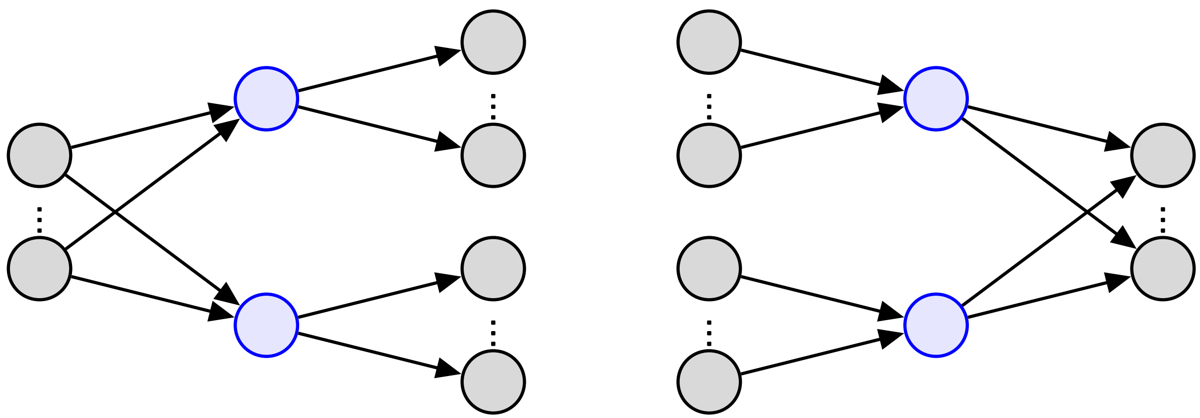
3. Cut an edge of T from each bad edge set

Exercise:
No path remaining in T corresponds to a nilpotent matrix
(any prefix of such a path is trivial or does not reduce)

$4 \to 5$: state amalgamations

Find a smaller representative in the conjugacy class of σ_T

Combine (amalgamate) two nodes if they 'behave the same'



same preimage

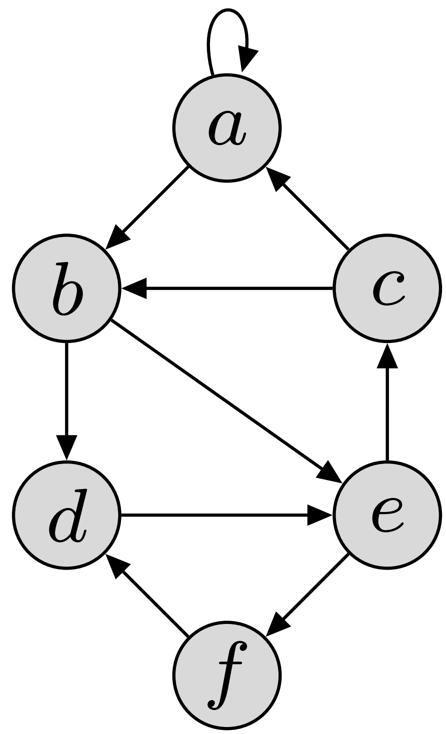
disjoint images

same image

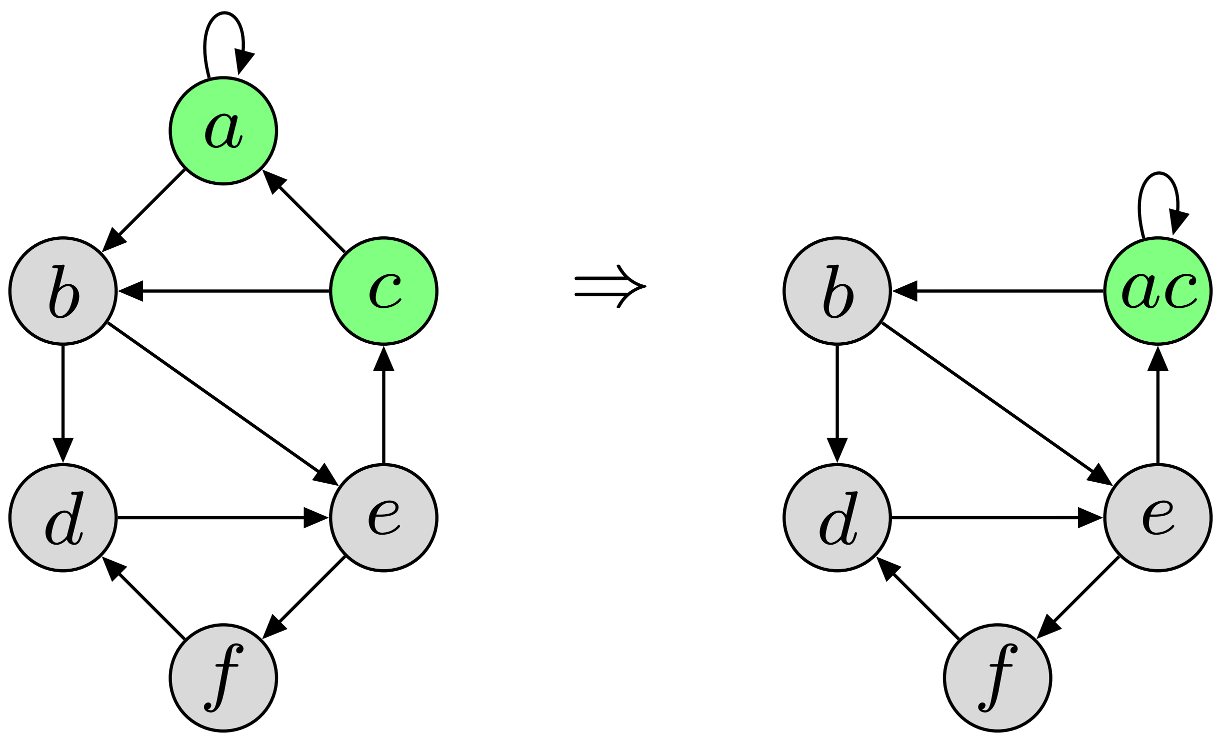
disjoint preimages

Finding smallest representative is NP-complete
(i.e., hard to compute)

Greedy algorithm fails:

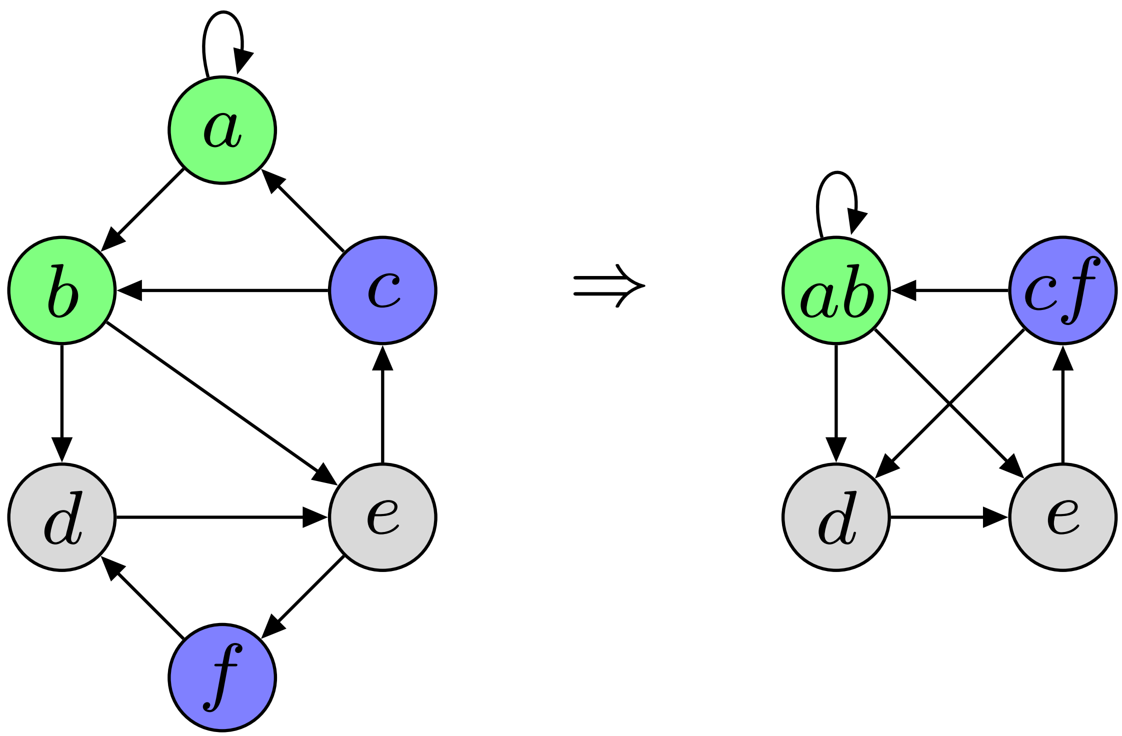


One choice of amalgamation



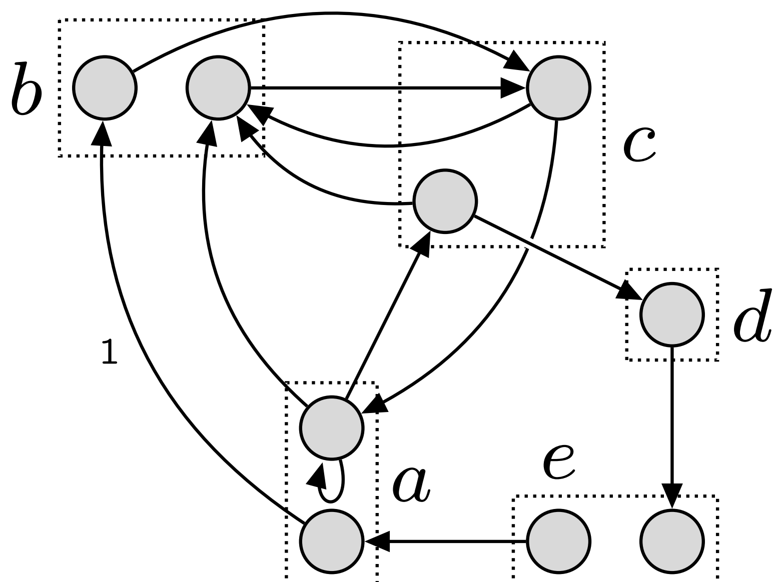
No further amalgamations

A better choice of amalgamation



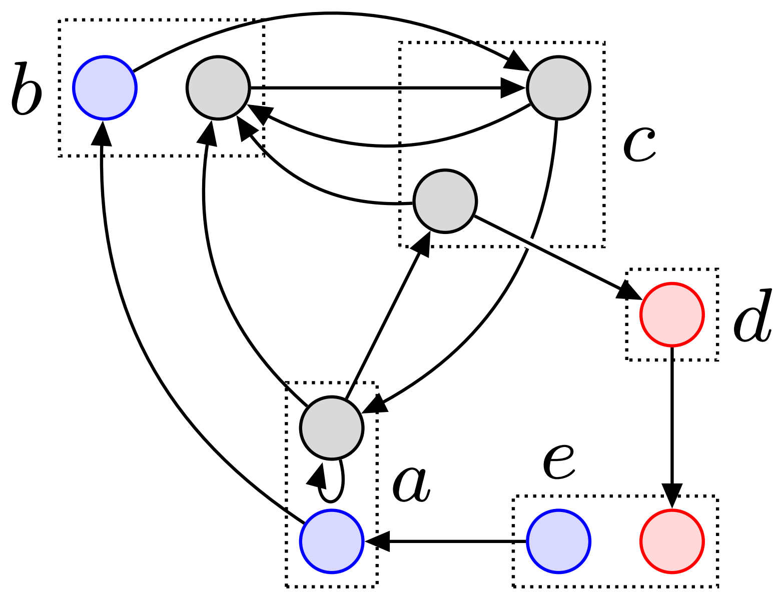
An example

Index map



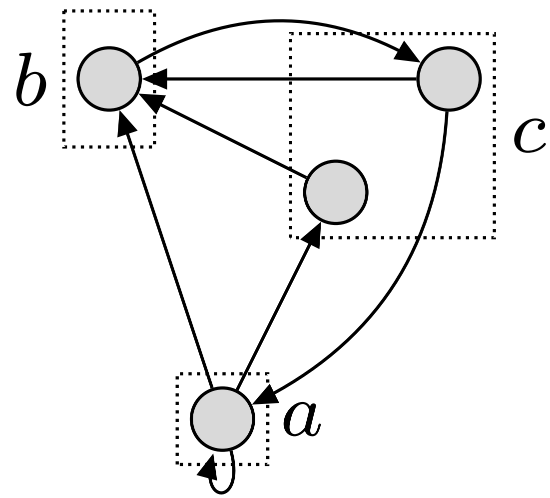
Possible cycle $a \rightarrow b \rightarrow c \rightarrow d \rightarrow e \rightarrow a$.

Index map with transient generators identified



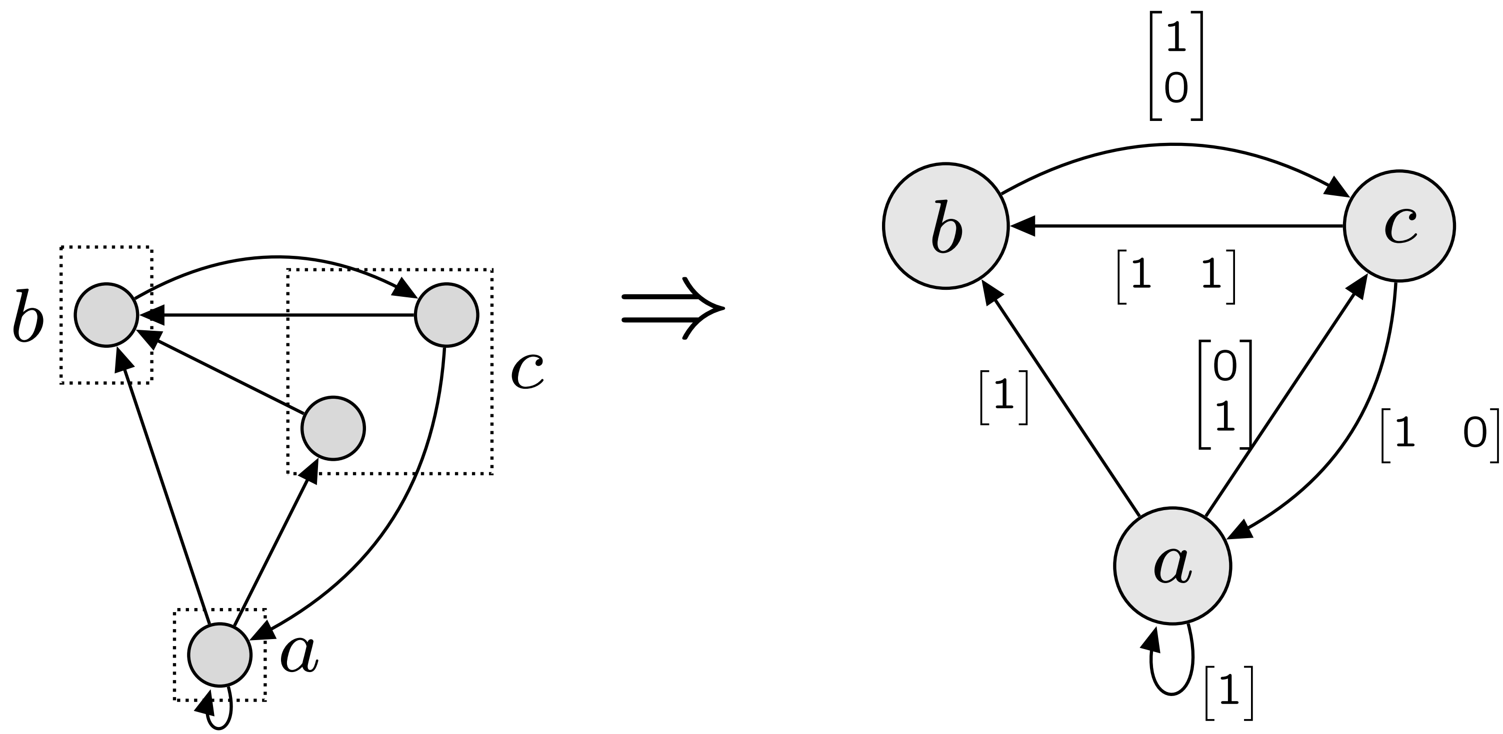
preimage generators in blue, image generators in red

Index map with transient generators removed

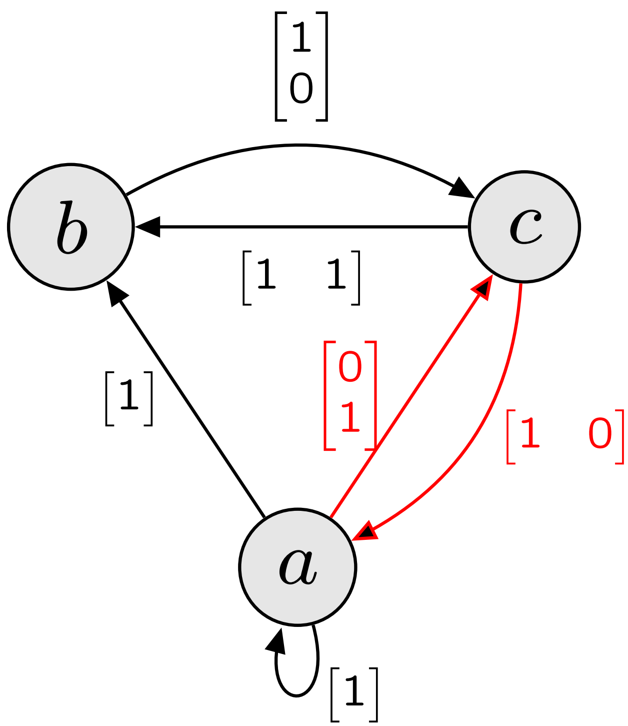


Unable to prove $a \to b \to c \to d \to e \to a$ cycle.

Map induced on regions of the phase space



An unverifiable cycle!

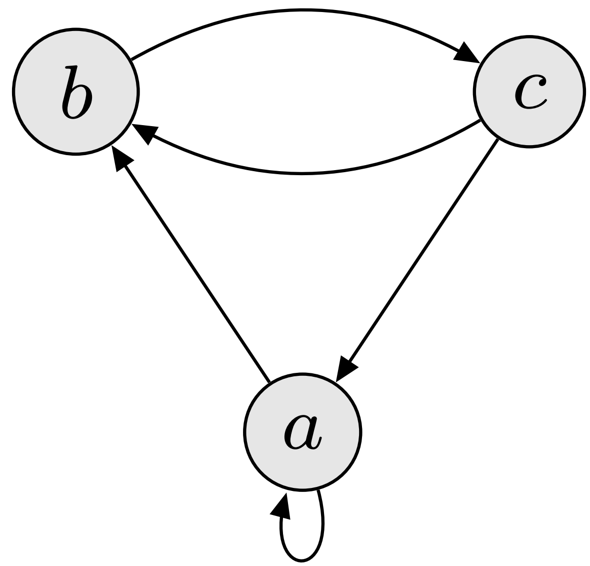


$$\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = 0$$

Edge set to cut: $\{(a,c),(c,a)\}$

Cut (a,c), since otherwise entropy is zero

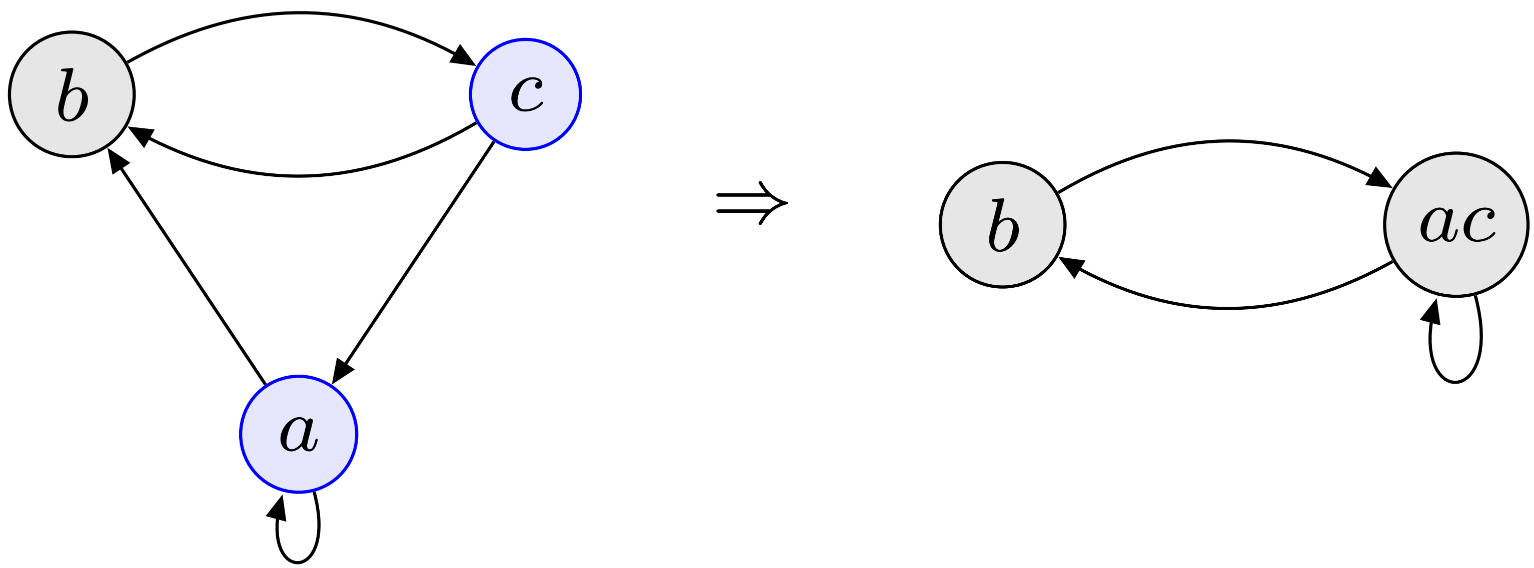
Resulting symbolic dynamical system



We now have a semi-conjugacy

Entropy of $f \geq$ entropy of $\sigma_T = 0.4812...$

Amalgamation!



a and c: same image and disjoint preimages

ERROR: slide index out of range

possible cause: talk is over