Minimax Option Pricing Meets Black-Scholes in the Limit

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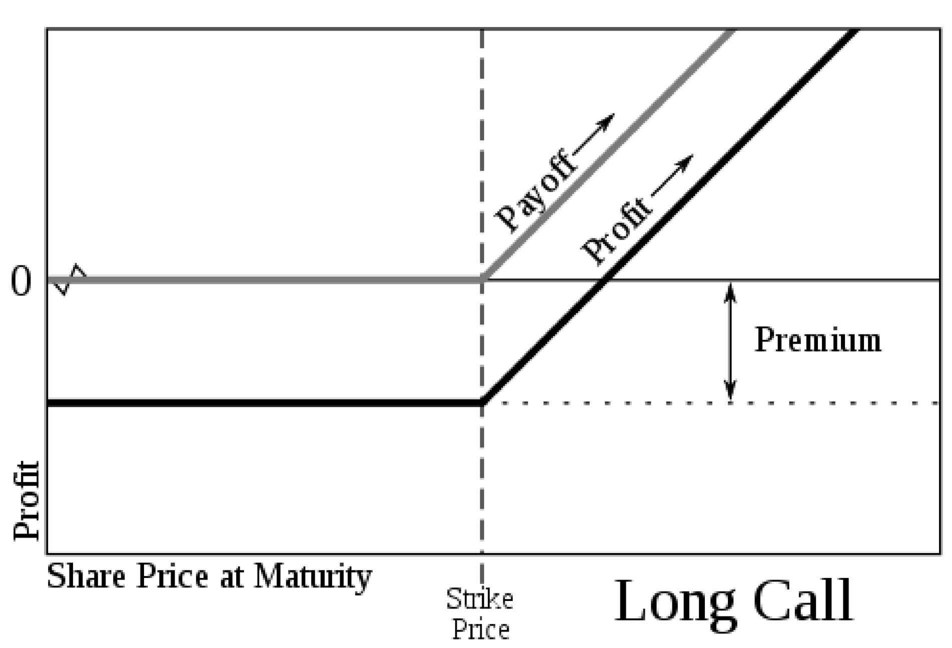
Objective

- Study an adversarial setting for pricing options
  - Zero-sum game: Nature vs. Investor
  - Option price is the value of the game
- Consider the limit as trade frequency increases
  - Under reasonable constraints, what does Nature's optimal strategy look like?
  - How does our price compare to the commonly-used Black-Scholes price?

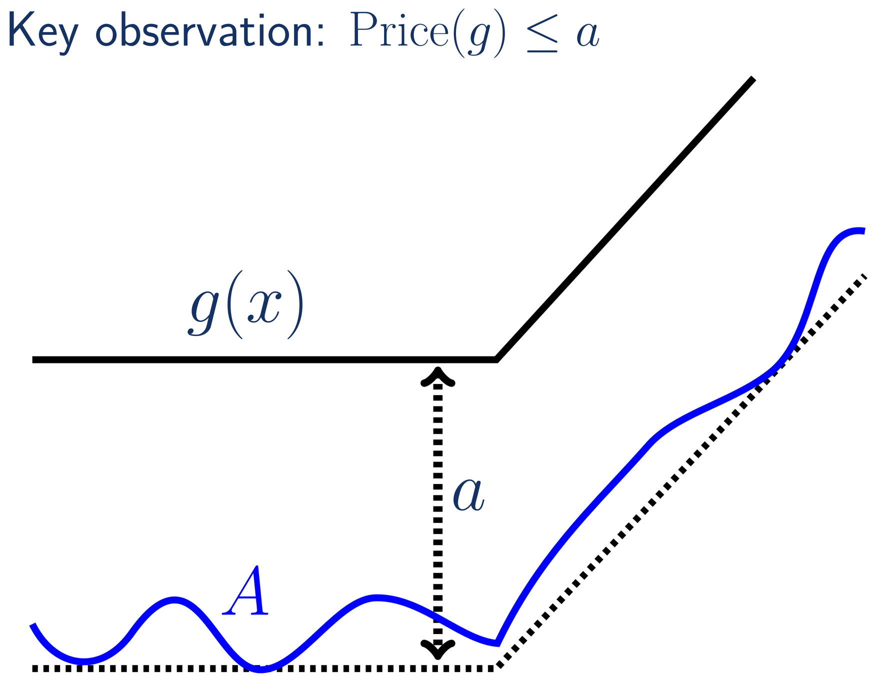
Options

- Types of option
  - European call/put - exercise @ expiration
  - American call/put - exercise any time
  - "Exotic" – basically any derivative
- Our setting
  - Exercised only at time $t = 1$
  - $X : [0,1] \to \mathbb{R}_+$ is the price path
  - Payoff is $g(X(1))$ for g *convex*
- European long call: $g(x) = \max(0, x - K)$



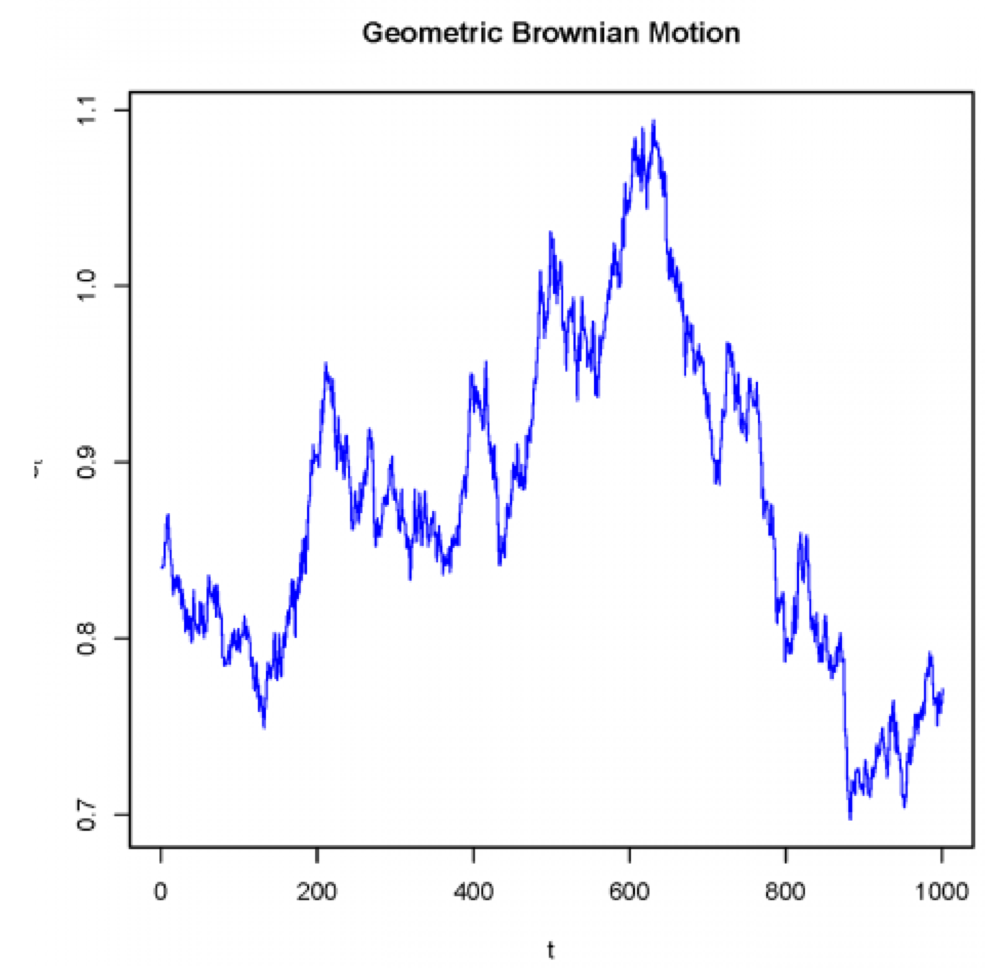
Replication Strategies

- Basic idea
  - a trading *algorithm* A with initial debt $\$a$
  - attempts to *replicate* the option g
  - payoff dominates the option, minus a
- Key observation: $\mathrm{Price}(g) \le a$



Black-Scholes

- Option Pricing under *Stochastic Assumptions*
  - Assume: Stock prices fluctuate according to Geometric Brownian Motion
  - Assume: We can trade continuously and buy/sell fractional shares



- Black-Scholes replication strategy
  - Let $V(S,t)$ be the value of the option at t
  - Let $\frac{\partial V}{\partial S}$ be the replaction portfolio
  - To solve for V, we need to solve a stochastic partial differential equation. Via Ito's Lemma, we get

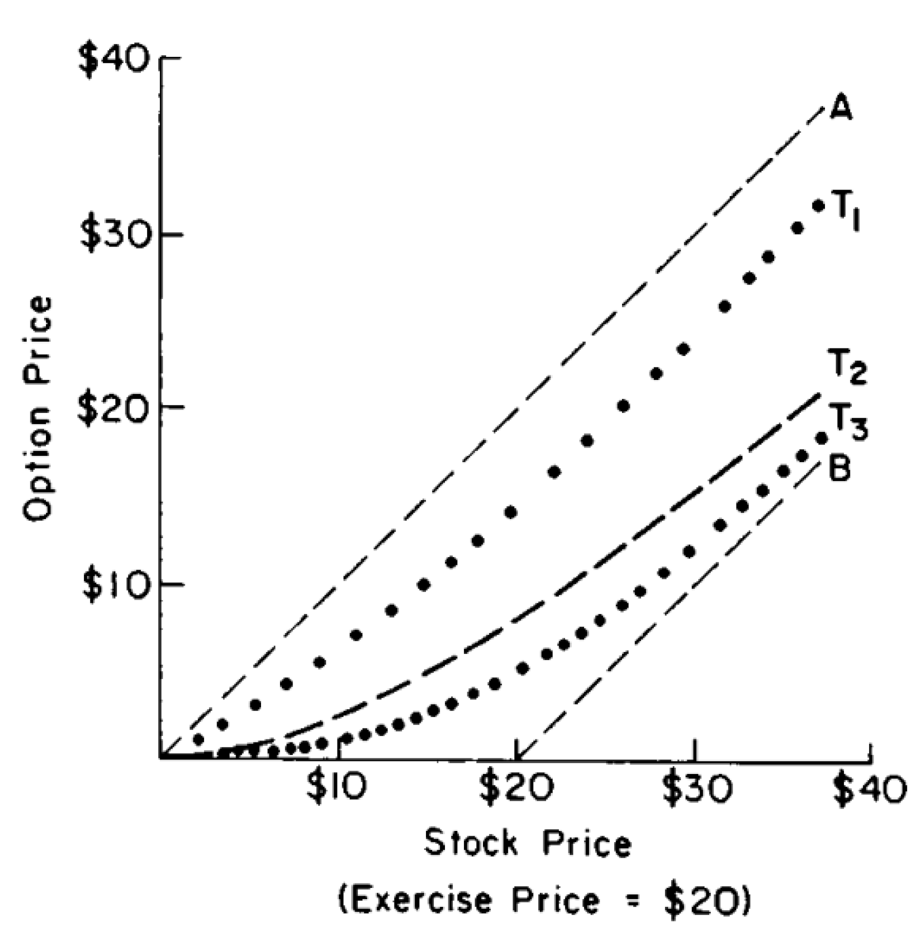
$$\frac{\partial V}{\partial S} - \frac{1}{2} S^2 \frac{\partial^2 V}{\partial S^2} = 0$$

  - Soution is:

$$V(S,t) = \mathbb{E}_{G\sim\mathsf{GBM}}[g(SG(1-t))]$$

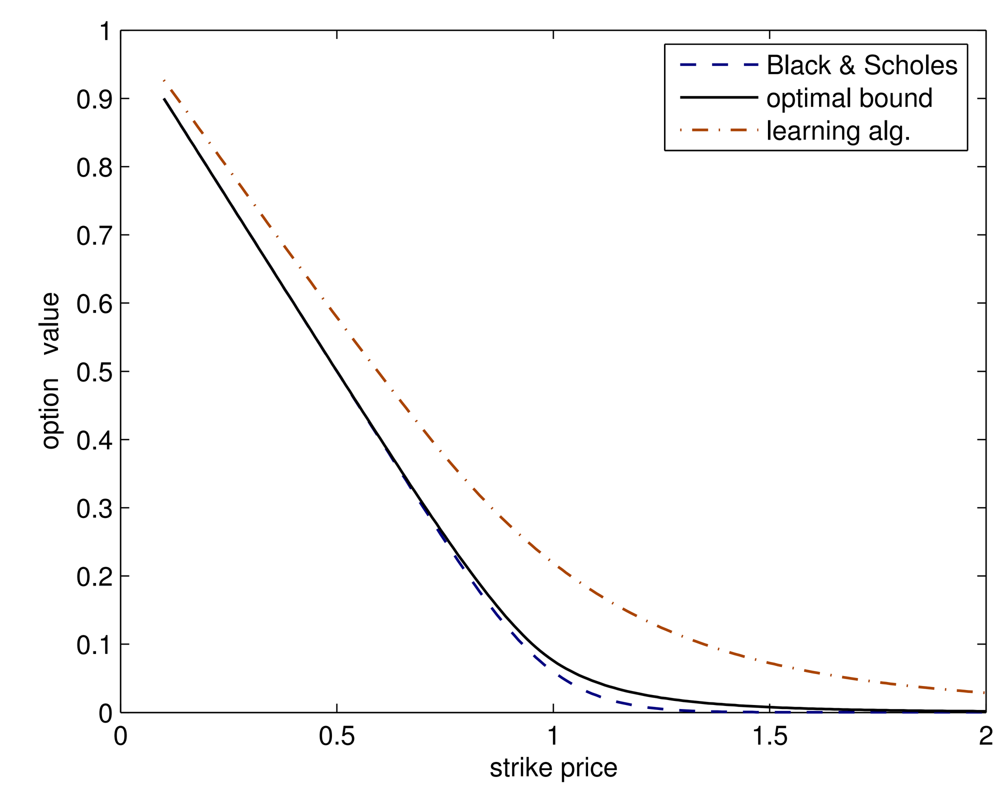
The Black-Scholes Price

$$\mathrm{Price}(g) = \mathbb{E}_{X\sim\mathsf{GBM}}[g(X(1))]$$



DeMarzo, Kremer, Mansour, 2006

- Option Pricing under *Adversarial Assumptions*
  - What if prices were chosen by an adversary?
  - Can we still replicate options in the worst case?
  - DeMarzo et al: **YES:** Construct replication strategies via exponential-weight style algorithm
  - Worst-case model gives a theoretical upper bound on the price of an option.



Our Game

- Nature vs. Investor

$$\inf_{A\in\mathcal{A}} \sup_{X\in\mathcal{X}} \mathbb{E}\left[g(X(1)) - \sum_m T_m \Delta_m\right]$$

  - Discrete-time trades at $t = m/n$, $m \in [n]$
  - $A \in \mathcal{A}$ is the replication algorithm (Investor) A chooses Δ_m to invest at time $t = m/n$
  - $X \in \mathcal{X}$ is the price path (Nature) T_m is the fluctuation at time $t = m/n$:

$$X\left(\tfrac{m}{n}\right) = X\left(\tfrac{m-1}{n}\right)(1 + T_m)$$

- Value of the game = option price!
- Interested continuous trading limit as $n \to \infty$

Constraints on Nature

- Bounded per-round variance:

$$\mathbb{E}\left[\left(\frac{X(t)}{X(s)} - 1\right)^2 \middle| X(s)\right] \le \exp(c(t-s)) - 1$$

  - (GBM satisfies this with equality)
  - In particular, $\mathbb{E}[T_m^2 | T_{m-1}] \le \exp(c/n) - 1$
- Bounded fluctuations:

$$|T_m| \le \zeta_n, \quad \text{for } \zeta_n \to 0$$

  - GBM does not satisfy this!
  - For a lower bound, we must truncate GBM
  - Need ζ_n to increase relative to trading freq:

$$\liminf_{n\to\infty} \frac{n\,\zeta_n^2}{\log n} > 16c$$

The Proof

Step I: Duality

- Sion's minimax theorem applies: $\inf \longleftrightarrow \sup$!

$$\sup_{X\in\mathcal{X}} \inf_{A\in\mathcal{A}} \mathbb{E}[g(X(1)) - \sum_m T_m \Delta_m]$$

- (By ζ_n fluctuation constraint, $\mathcal{X}$ is compact)

Step II: Martingality

- Claim: $T_1, \ldots, T_m$ is a martingale sequence
- Suppose not; then Investor can make some $T_m \Delta_m \to \infty$
- Value of the game is now $\sup_{\substack{X\in\mathcal{X}\\ \{T_m\}\text{mtg}}} \mathbb{E}[g(X(1))]$

Step III: Max Variance

- Since g is convex, T_m has maximal conditional variance for all m!
- Hence, $\mathbb{E}[T_m^2 \mid T_{m-1}] = \exp(c/n) - 1$

Step IV: Finale

- Let X_n^* be the optimal price path for n trades
- Theorem: $X_n^* \xrightarrow{d}$ GBM as $n \to \infty$
- Corollary: $\lim_{n\to\infty} \mathbb{E}[g(X_n^*(1))] = \mathbb{E}[g(\mathsf{GBM}(1))]$
- Option price approaches Black-Scholes price!

Conclusion

- The Black-Scholes pricing scheme is valid even in an adversarial model!
- Moreover, the stochastic assumption made by the Black and Scholes can be derived as the optimal strategy of Nature in our model.

Future Work

- Allowing price jumps
  - Only consider points $X(1/n) \ldots X(n/n)$
- Per-round variance $\to$ variance budget

$$\sum_{m=1}^{n} \mathbb{E}[T_m^2 \mid T_{m-1}] \le c$$

  - Theorem breaks: $X_n^* \not\to$ GBM
  - But final price converges: $X_n^*(1) \to \mathsf{GBM}(1)$
  - Hence we still have our Corollary:

$$\lim_{n\to\infty} \mathbb{E}[g(X_n^*(1))] = \mathbb{E}[g(\mathsf{GBM}(1))]$$

  - We still obtain the Black-Scholes price!