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Mechanism Design

Properties 000000

## General Characterizations of Truthfulness via Convex Analysis

## Rafael Frongillo

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Joint work with Ian Kash (MSRC)

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## Ian Says Hi!



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## Warm-up: Convex Functions



#### Definition

 $G : \mathcal{T} \rightarrow \mathbb{R}$  is *convex* if for all  $x, y \in \mathcal{T}$  and all  $\alpha \in [0, 1]$ 

 $\alpha G(x) + (1-\alpha)G(y) \geq G(\alpha x + (1-\alpha)y)$ 

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## Warm-up: Convex Functions



#### Definition

A linear function  $dG_t : \mathcal{T} \rightarrow \mathbb{R}$  is a *subgradient* to *G* at *t* if

 $\forall t' \in \mathcal{T} \quad G(t') \ge G(t) + dG_t(t'-t)$ 

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## Pointwise Supremum



## Fact

#### If $G_i$ are convex functions for $i \in I$ , then G is convex:

$$G(t) := \sup_{i \in I} G_i(t)$$

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Mechanism D	esian		

Single-player mechanism:

- Outcome space *O* possible allocations
- Type space  $\mathcal{T} = (\mathcal{O} \rightarrow \mathbb{R})$  valuation functions
- Allocation rule  $a : T \rightarrow O$  reports to outcomes
- Payment rule  $p : T \rightarrow \mathbb{R}$  reports to payments

Bidder with type t who reports  $t' \in \mathcal{T}$  has net utility

U(t',t) = t(a(t')) - p(t')

Truthfulness condition

 $\forall t, t' \in \mathcal{T} \ U(t', t) \leq U(t, t)$ 

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Mverson 1981			

For single-parameter mechanisms:

Theorem

a is implementable  $\iff$  a is monotone

Implementable means payments p making (a, p) truthful

Equivalently:

Theorem

a is implementable  $\iff \exists G : \mathcal{T} \rightarrow \mathbb{R}$  convex s.t. a is a subgradient to G

G is the consumer surplus

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Myerson 1981			

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Scoring Rules			

- Outcome space *O* mutually exclusive events
- Private belief  $p \in \Delta_{\mathcal{O}}$  probabilities over outcomes
- Scoring rule  $S : \Delta_{\mathcal{O}} \times \mathcal{O} \rightarrow \mathbb{R}$  score of report given an outcome

Expected score of report p' given truth p is

$$S(p',p) := \mathop{\mathbb{E}}_{o \sim p} \left[ S(p',o) \right]$$

$$\forall p, p' \ S(p', p) \leq S(p, p)$$

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## Gneiting and Raftery 2007

#### Theorem

Scoring rule S is truthful  $\iff$  there is some convex  $G : \Delta_{\mathcal{O}} \rightarrow \mathbb{R}$ with subgradients  $\{ dG_p \}$  such that

$$S(p, o) = G(p) + dG_p (\mathbf{1}_o - p)$$

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## What's the Connection?

Mechanism:

- Outcomes *O*
- Type  $\mathcal{T} = (\mathcal{O} \rightarrow \mathbb{R})$
- Utility U(t', t)

Scoring rule:

- Outcomes O
- Belief  $p \in \Delta_{\mathcal{O}}$
- Score S(p', p)

 $\frac{\text{Truthfulness}}{U(t',t) \le U(t,t)}$ 

Truthfulness  $S(p', p) \le S(p, p)$ 

 $U(t',t) = t(a(t')) - p(t') \qquad S(p',p) := \mathop{\mathbb{E}}_{o \sim p} \left[ S(p',o) \right]$  $= \langle t, \mathbf{1}_{a(t')} \rangle - p(t') \qquad = \langle p, S(p', \cdot) \rangle$ 

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What's the	Connection?		

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What's	the Connection?		
Mech	anism:	Scoring rule:	
	Dutcomes $\mathcal O$	Outcomes O	
<b>–</b> 7	$Fype \ \mathcal{T} = (\mathcal{O} \to \mathbb{R})$	Belief $p \in \Delta_{\mathcal{O}}$	
🔳 L	Jtility U(t', t)	Score $S(p', p)$	
	Truthfulness	Truthfulness	<b>;</b>
l	$U(t',t) \leq U(t,t)$	$S(p',p) \leq S(p)$	, p)

 $U(t',t) = t(a(t')) - p(t') \qquad S(p',p) := \mathop{\mathbb{E}}_{o \sim p} \left[ S(p',o) \right]$  $= \langle t, \mathbf{1}_{o(t')} \rangle - p(t') \qquad = \langle p, S(p', \cdot) \rangle$ 

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V	What's the Connection?						
	Mechanism:	Scoring rule:					
	Outcomes O	Outcomes O					
	Type $\mathcal{T} = (\mathcal{O} \to \mathbb{R})$	Belief $p \in \Delta_{\mathcal{O}}$					
	Utility $U(t', t)$	Score $S(p', p)$					
	Truthfulness	Truthfulness					
	$U(t',t) \leq U(t,t)$	$S(p',p) \leq S(p,p)$					
	U(t',t) = t(a(t')) - p(t')	$S(p',p) := \mathop{\mathbb{E}}_{o \sim p} \left[ S(p',o) \right]$					

Backg	round oo	Main Result ●ooooo		Mechanism Design 0000	Properties		
Wł	What's the Connection?						
	Mechanism: Outcomes $C$ Type $T = (C$ Utility $U(t', t')$	$\mathcal{O} \rightarrow \mathbb{R}$ )	Sc	oring rule: Outcomes $\mathcal{O}$ Belief $p \in \Delta_{\mathcal{O}}$ Score $S(p', p)$			
	$\frac{\text{Truthfuln}}{U(t',t) \leq U}$	ess J(t, t)		$\frac{\text{Truthfulness}}{S(p',p) \le S(p,p)}$	))		
	$U(t',t) = t(a(t') = \langle t, 1_{d} \rangle$	f(t)) - p(t') $p_{a(t')} \rangle - p(t')$	S	$S(p',p) := \mathop{\mathbb{E}}_{o \sim p} \left[ S(p') = \langle p, S(p') \rangle \right]$	ɔ′, o)] ′, ·) ⟩		

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## Our Model: Affine Score

- **Type space** T any subset of a vector space
- Reward space  $\mathcal{A} \subseteq \operatorname{Aff}(\mathcal{T} \to \mathbb{R})$  affine functions on types
- Affine score  $S : \mathcal{T} \rightarrow \mathcal{A}$

#### Truthfulness condition

## $S(t')(t) \leq S(t)(t)$

Observation:  $G(t) := \sup_{t'} S(t')(t)$  convex and S truthful  $\implies G(t) = S(t)(t)$ 

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Our Model: A	Affine Score		

## - - . .

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## A General Truthfulness Characterization

#### Theorem

Affine score  $S : T \to A$  is truthful if and only if there exists some convex  $G : Conv(T) \to \mathbb{R}$ , and subgradients  $\{dG_t\}$ , such that

$$S(t')(t) = G(t') + dG_{t'}(t - t').$$

# Techniques from Gneiting-Raftery and Archer-Kleinberg Immediately gives previous scoring rule and mechanism characterizations

Background	Main Result oooeoo	Mechanism Design	Properties
Proof: Con	lex ${\cal T}$		

## Proof of $\Leftarrow$ : $S(t')(t) = G(t') + dG_{t'}(t - t')$ $\leq G(t) = S(t, t)$ by def. subgradient

#### Proof of $\implies$ :

 $G(t) := \sup_{t \in S} S(t')(t)$  convex as pointwise supremumb

Define  $dG_t(\cdot) = S_t(t)(\cdot)$  linear part of S(t)

 $S(t)(\cdot)$  subgradient to G at t: by truthfulness

Background	Main Result ooo●oo	Mechanism Design	Properties
Proof: Con	Vex $\mathcal{T}$		

Proof of 
$$\Leftarrow$$
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$$S(t')(t) = G(t') + dG_{t'}(t - t')$$

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Proof of  $\implies$ :

•  $G(t) := \sup_{t'} S(t')(t)$  convex as pointwise supremum!

- Define  $dG_t(\cdot) = S_l(t)(\cdot)$  linear part of S(t)
- **S(t)(\cdot) subgradient to G at t:** by truthfulness

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Proof: Con	Vex $\mathcal{T}$		

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Background 000000	Main Result	Mechanism Design	Properties
Proof: Non-Cor	nvex ${\cal T}$		

#### Proof of $\Leftarrow$ : same.

Proof of  $\implies$ :

Gonsider  $\hat{t} \in Conv(\mathcal{T}) \setminus \mathcal{T}$ 

• Write  $\hat{t} = \sum_{l} \alpha_{l} t_{l}$  for  $t_{l} \in \mathcal{T}$ 

Define  $S(t)(\hat{t}) = \sum_{l} \alpha_{l} S(t)(t_{l})$ 

Define  $G(\hat{t}) = \sup_{t \in T} S(t)(\hat{t})$ 

Background	Main Result oooooo	Mechanism Design	Properties
Proof: Non-Cor	nvex ${\cal T}$		

Proof of  $\Leftarrow$ : same.

Proof of  $\implies$ :

Consider  $\hat{t} \in Conv(\mathcal{T}) \setminus \mathcal{T}$ Write  $\hat{t} = \sum_i \alpha_i t_i$  for  $t_i \in \mathcal{T}$ Define  $S(t)(\hat{t}) = \sum_i \alpha_i S(t)(t_i)$ Define  $G(\hat{t}) = \sup_{t \in \mathcal{T}} S(t)(\hat{t})$ 

Background	Main Result ooooo●o	Mechanism Design	Properties
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Background	Main Result ooooeo	Mechanism Design	Properties
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Immediate New	/ Results		

1 Proper scoring rules for non-convex sets of distributions

*Fewer constraints*  $\implies$  *more scoring rules?* 



2 "Local" mechanisms and scoring rules Convexity is a local property

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Immediate New	/ Results		

1 Proper scoring rules for non-convex sets of distributions

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2 "Local" mechanisms and scoring rules Convexity is a local property

Background	Main Result	Mechanism Design ●০০০	Properties
Mechanism De	sign: Implemen	tability of a	

#### Definition

 $\{ dG_t \}_{t \in \mathcal{T}} \text{ satisfies } cyclic \text{ monotonicity (CMON) if for all finite sets} \\ \{ t_0, \dots, t_k \} \subseteq \mathcal{T}, \\ \sum_{i=0}^k dG_{t_i}(t_{i+1} - t_i) \leq 0.$ 

CMON with k = 2 is Weak monotonicity (WMON).

Let 
$$L_{xy} = \int_0^1 dG_{\beta y+(1-\beta)x} (y-x) d\beta.$$

#### Definition

 $\{dG_t\}_{t\in\mathcal{T}}$  satisfies path independence (PI) if for all  $x, y, z \in \mathcal{T}$ 

$$L_{xy} + L_{yz} = L_{xz}$$









Background 000000	Main Result	Mechanism Design ooo●	Properties
Reproving Müll	er et al.		



#### New proof via construction of G:

## ■ Fix *G*(*t*<sub>0</sub>)

- Extend  $G(t) = L_{t_0 t}$  integrable by WMON, consistent by PI
- Subgradient by simple computation

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Reproving Müll	er et al.		



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## Q: What if types are exponential (or infinite!) in size?

A: Use summary information / low-dim representation

Examples:

- Scoring rules for statistics [Lamber]
  - Rankings instead of utilities [

[Lambert-Pennock-Shoham, Gneitir [Carroll]



## Q: What if types are exponential (or infinite!) in size?

## A: Use summary information / low-dim representation

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- Scoring rules for statistics [Lambert-Pennock-Shoham, Gneiting]
- Rankings instead of utilities [C]

[Lambert-Pennock-Shoham, Gnei [Carroll]



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- Rankings instead of utilities [Carroll]

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#### Wish to change report space from T to some other R

$$S: R \to Aff(\mathcal{T} \to \mathbb{R}); \qquad S(r)(t)$$

What does truthful mean now?

Background	

## Definition

**Properties** 

A *property* is a map  $\Gamma : \mathcal{T} \to R$  specifying the correct report  $r = \Gamma(t)$  for each type *t*.

#### Truthfulness condition

## $S(r')(t) \leq S(\Gamma(t))(t)$

We say *S elicits* Г.

Background	

#### Definition

**Properties** 

A *property* is a map  $\Gamma : \mathcal{T} \to R$  specifying the correct report  $r = \Gamma(t)$  for each type *t*.

Truthfulness condition

 $S(r')(t) \leq S(\Gamma(t))(t)$ 

We say S *elicits*  $\Gamma$ .

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## A New Result

#### Theorem

Property  $\Gamma$  is elicitable iff there exists  $G : \mathcal{T} \to \mathbb{R}$  differentiable and convex, and map  $\varphi : R \to \nabla G(\mathcal{T})$ , such that  $\varphi(\Gamma(t)) = \nabla G(t)$ .

## New insights:

- Elicitable properties == subgradients!
- Properties specify where *G* should be *flat*

## A New Result

## Theorem

Property  $\Gamma$  is elicitable iff there exists  $G : \mathcal{T} \to \mathbb{R}$  differentiable and convex, and map  $\varphi : R \to \nabla G(\mathcal{T})$ , such that  $\varphi(\Gamma(t)) = \nabla G(t)$ .

## New insights:

- Elicitable properties == subgradients!
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## Finite R: Power Diagram



Cells = types with same report. Application: rankings!

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## Thanks!